Using Computers to Compute

Calculating Sine and Cosine Using CORDIC

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Thursday, July 26th, 2018

Outline

Overview

I will talk briefly about the following topics:

- What are sine and cosine?
- How do humans do math?
- How do simple computers do math?
- What is CORDIC?
- How does CORDIC work?
  - Including graphic examples

Why?

- It's fun
- Tau Day (6/28). Last month's meeting was a social, so this talk is late
- History of computing
- Math! Who doesn't like learning how to do math calculations?
  - Or at least having a computer spare us the work?

Tau Manifesto Plug

- If you haven't read Michael Hartl's Tau Manifesto, you should.
- tauday.com
- Makes understanding trigonometry much easier

Sine and Cosine

- What are they?
- **Tri**gonometry. Supposedly about **Tri**angles.
- It's really about circles.

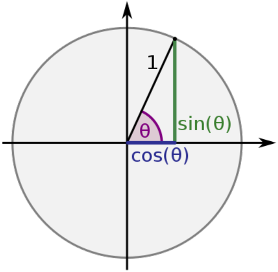
Sine and Cosine Illustrated

Unit Circle: Circle of radius 1, centered at the origin, (0,0)

Theta (θ): Angle pointing to point on a circle

Sine: Vertical coordinate of point

Cosine: Horizontal coordinate of point

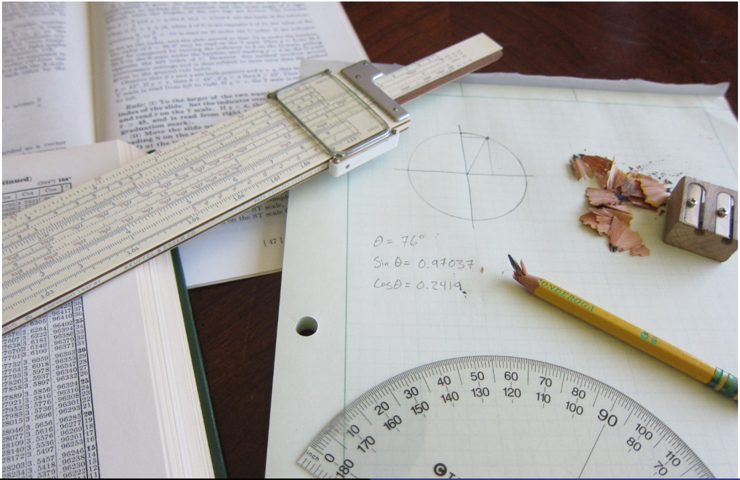


Wikipedia

Why are these functions important?

- Fundamental math
- Used in very many engineering calculations
- Astronomy, Surveying, Navigation, Mechanics, Radio Engineering, Signal Processing
- Converting from rectangular to polar coordinates
- I won't even try to list all the uses

Computing before digital computers



Tools of the Trade

- Table books (printed spreadsheet)
- Slide rule (fancy lookup table)
- Pencil, paper
- Maybe even a protractor (for minimum accuracy)
- Did you spot the error in the previous slide?
  - $sin\theta$ should have been 0.97029. Oops, I copied the wrong table value.

How are the tables made?

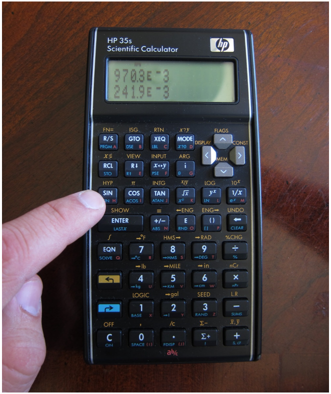
- By hand, with Taylor series like this:

$$sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

$$cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

- That's a lot of multiplications!
  - Even after clever optimizations
  - We'll get back to that in a minute

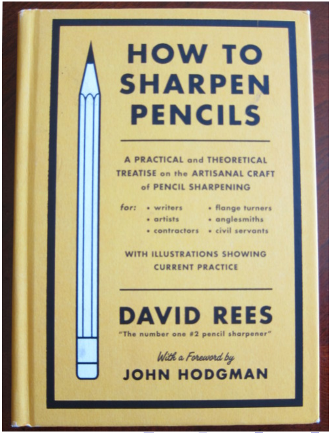
There has to be a better way!



AS SEEN ON TV

Quick side-note for the unlucky

- If you **are** stuck working with pencil and paper, this book has you covered.



Calculators

- Okay, so we have calculators.
- But how do they work?

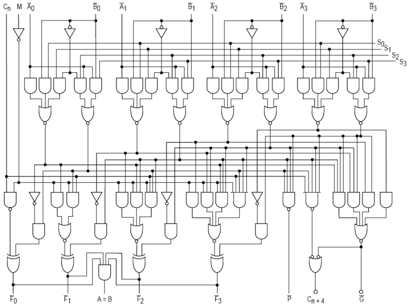
Digital logic for math

- Digital circuits exist to quickly calculate basic arithmetic operations
- Addition, subtraction, bitshift, etc.
- Bitshift is multiplication and division by powers of two only
- Arithmetic Logic Unit (ALU)

Digital logic for math

- Simple computers have no multiplication or division operators
- Why? Multiplication takes multiple steps (more time) and more transistors
- Division takes even more

A simple ALU



Wikipedia

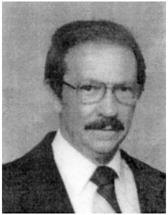
- 4-bit ALU schematic, simplified
- Look at this thing
- You don't want to make this even more complicated by building a multiplier too, do you?

Sine and cosine without multiply

- Traditional methods of calculating sine and cosine require slow and expensive multiply circuits
- Can we calculate sine and cosine without multiply circuits?

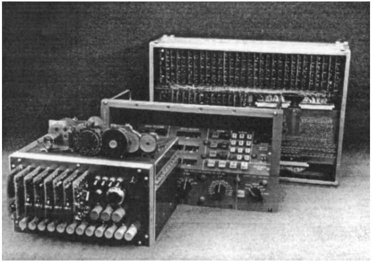
CORDIC

- **CO**ordinate **R**otation **DI**gital **C**omputer
- Jack E. Volder, The CORDIC Trigonometric Computing Technique, IRE Transactions on Electronic Computers, September 1959.



The CORDIC I Computer

- Special purpose computer to demonstrate algorithm
- 96.5773 kHz clock frequency
- Built in 1960
- Led to CORDIC II built for the Air Force and B-58 bomber
- Replaced analog computers



Journal of VLSI Signal Processing

The CORDIC benefits

- Uses only:
  - Addition
  - Subtraction
  - Bitshift
  - A small table look up
- No multiplications!
- Can be built with the simple ALU circuits

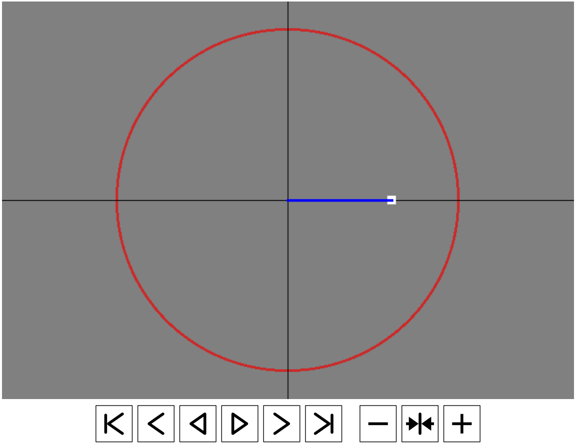
How was it discovered?

- Trigonometric identities:
  - Collections of mathematical facts collected over thousands of years
- Not obviously useful at first look, until you see one used well
- This is an example
- *"Necessity is the mother of invention."* –Jack Volder

An apology

- I'm sorry, but I didn't have time to create images for the following slides.
- We will all have to use our imaginations, or study this part later.
- There are animated examples at the end.

Sneak peek



Quick derivation of CORDIC 1

- Suppose you already know the coordinates of one point on the circle with angle α
  - sin(α) and cos(α)
- What about another point a small angle away?
  - sin(α+θ) and cos(α+θ)

$$sin(\alpha + \theta) = cos(\alpha)cos(\theta) - sin(\alpha)sin(\theta)$$

$$cos(\alpha + \theta) = sin(\alpha)cos(\theta) + cos(\alpha)sin(\theta)$$

- These equations rotate a known point around the circle

Quick derivation of CORDIC 2

Coordinate Rotation:

$$sin(\alpha + \theta) = cos(\alpha)cos(\theta) - sin(\alpha)sin(\theta)$$

$$cos(\alpha + \theta) = sin(\alpha)cos(\theta) + cos(\alpha)sin(\theta)$$

Or:

$$x_{next} = x_{in}cos(\theta) - y_{in}sin(\theta)$$

$$y_{next} = y_{in}cos(\theta) + x_{in}sin(\theta)$$

In code:

```
x_next = x_in * cos(theta) - y_in * sin(theta)
y_next = y_in * cos(theta) + x_in * sin(theta)
```

Matrix notation

- Graphics programmers might recognize this as an image rotation matrix

$$P_{next} = \begin{bmatrix} x_{next} \\ y_{next} \end{bmatrix} = A * P_{in}^{T} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix}$$

Quick derivation of CORDIC 3

Math trick: divide both sides of the equation by $cos(\theta)$.

$$x_{next}/cos(\theta) = x_{in} - y_{in}tan(\theta)$$

$$y_{next}/cos(\theta) = y_{in} + x_{in}tan(\theta)$$

In code:

```
x_next = x_in - y_in * tan(theta)
y_next = y_in + x_in * tan(theta)
```

`x_next` and `y_next` grow each time we do this (since cos(x)<1), but we can fix that later.

Where does this get us?

- So now we can easily add a fixed angle to any sine/cosine result and get a new point
  - If we know the arctangent of that angle
- We can move to any angle/point by repeatedly adding a small angle (say, one degree) to a starting point, until we have the angle we want
- Or, we can be smart about our increments: start big, and get smaller as we zero in on the answer

Binary search for angles

- Idea: start with a quarter rotation, then an eighth, a sixteenth, etc.
- Binary search!
- Need to be able to rotate both forward and backwards (we can)
- So, when our angle sum is higher than our target angle, rotate clockwise.

Final trick

- Choose the angle of our successive jumps so that $tan(\theta)$ is always a power of two

$$x_{next}/cos(\theta) = x_{in} - y_{in} * 2^i$$
$$y_{next}/cos(\theta) = y_{in} + x_{in} * 2^i$$

In code:

```
x_next = x_in - (y_in >> i)
y_next = y_in + (x_in >> i)
```

Cleanup

- Our point's distance from the origin grows each iteration
- Start with a point inside the unit circle $(K, 0)$
- It will end up on the unit circle at the end
- I won't show the math for calculating K, but it isn't complicated

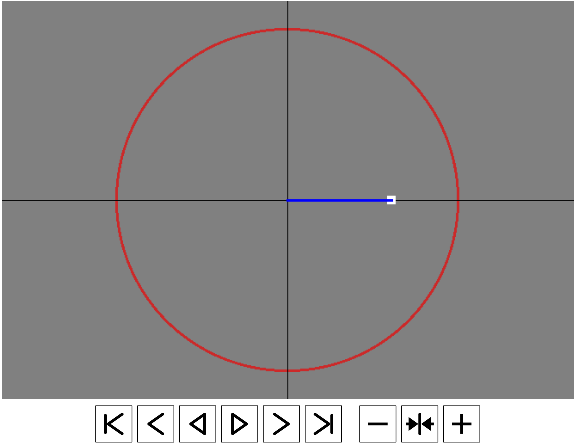
8-bit CORDIC code

```
/* 8-bit CORDIC algorithm for angles in first quadrant. */
int8_point_t do_cordic_8 (uint8_t binangle) {
  /* Calculate sin, cos using CORDIC */
  /* Initialize values */
  int8_t x = K_8;
  int8_t y = 0;
  int8_t z = binangle; /* Error term */

  for (int i = 0; i < 8; i++) {
    int8_t next_x, next_y, next_z;
    if (z >= 0) {
      next_x = x - (y >> i);
      next_y = y + (x >> i);
      next_z = z - arctan_8[i];
    }
    else {
      next_x = x + (y >> i);
      next_y = y - (x >> i);
      next_z = z + arctan_8[i];
    }

    x = next_x;
    y = next_y;
    z = next_z;
  }

  int8_point_t result;
  result.x = x;
  result.y = y;

  return result;
}

const uint8_t arctan_8[8] = {
  32, 19, 10, 5, 3, 1, 1, 0
};

const int8_t K_8 = 78-1;
```

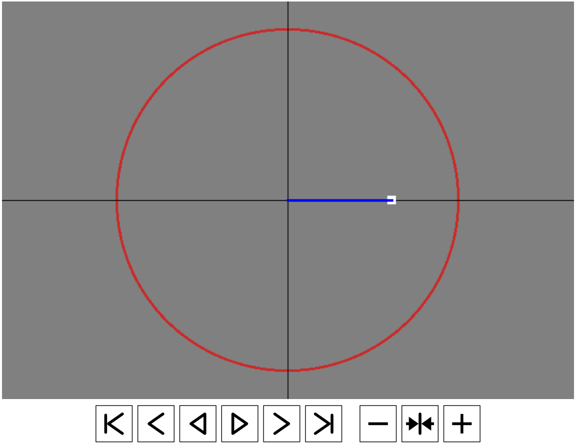
Examples

- The following examples show the CORDIC algorithm seeking the correct result
- In eight steps it gains eight bits of precision towards the correct answer (0.4% max error)
- Created with Dr. Geo geometry software
  - Uses Smalltalk language for scripting geometric figures

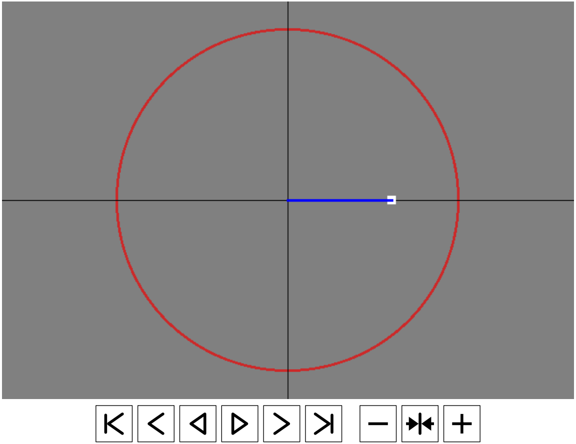
One radian



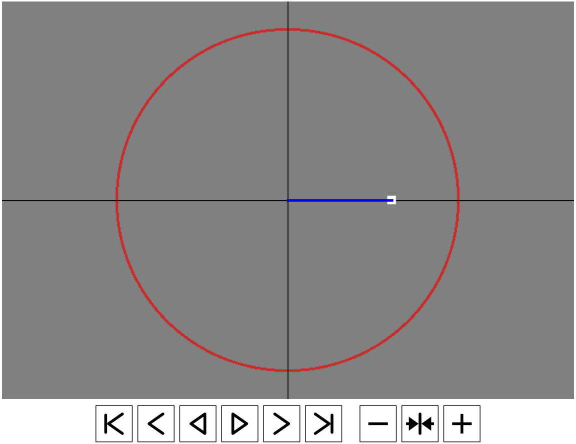
One half radian



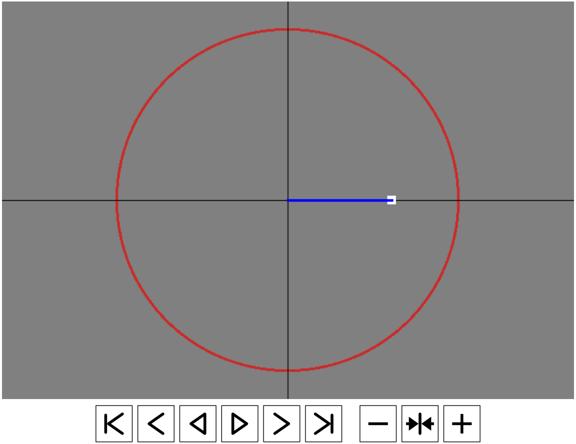
Two radians



Zero radians



90 degrees



Any Questions?

- Want to know more?